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Objective causality

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Abstract. Any change of state can be represented objectively and causally in the form of a transformation $T: W \to w = T(W) = \Sigma_j E_j W E_j^\dagger$, where $\{E_j\}$ are operators of the object system itself, and define the cause of the evolution from the initial state W to the final state w. Three axiomatically simple criteria are proposed in order to define that class of transformations $M(A)$ which accomplish the measurement of a given property $A = \Sigma_n A_n P_n$. This class turns out to hold an infinity of equivalent transformations, which all perform the same measurement, but employ distinct causes $\{E_j\}$. One such transformation is the projection $w = \Sigma_n P_n W P_n$ which is traditionally held to *define* the measurement of A. While our results are therefore consistent with standard theory, the wider class we consider includes transformations which accomplish the measurement by means of physical causes, such as rotations, translations, etc. A theorem is proved which shows the following. (a) For every measurement $T \in M(A)$ the cause must be compatible with A, i.e. $[A, E_j] = 0$ for all members of $\{E_j\}$. This means that the operators of quantum theory define the very operations which cause the measurement of the physical properties they represent. (b) To achieve a measurement a certain design criterion must be satisfied by the cause $\{E_j\}$. (c) The measured statistic $\{w_n\}$ always agrees with the standard probability prediction made within quantum theory on the basis of the initial object state W. If $w = \Sigma_n w_n P_n$ is the result of a measurement of A, and $P(n) = \mathrm{Tr}(WP_n)$ is the predicted probability that its value is A_n, then for every $T \in M(A)$: $w_n = P(n)$ for all n. This provides the empirical foundation for the quantal probability calculus, without resort to the traditional projection postulate. Our criteria which make this feasible rely exclusively on empirical principles based on classical logic—in an operational context. Several examples are investigated, including measurements performed by so-called operations. It is demonstrated that under certain circumstances an apparatus (not necessarily macroscopic) can both perform the measurement and record its outcome.

1. Introduction

The present work deals with so-called complete measurements and with the probabilistic/statistical interpretation of quantum theory. In a subsequent paper the theory will be generalised to incomplete measurements, whereby such concepts as indeterminacy and uncertainty can be rigorously investigated. A brief and informal account of some of these results has been given elsewhere (Larsen 1986b). Along with the formal structure we take care to supply remarks which aim to clarify the physical meaning of the theoretical concepts and of the results that are obtained. The main result is theorem 4.1, whose significance is explained in the remarks appended to it, and in § 6.

2. Preamble

Definition 2.1 (quantum principle). Let $\mathbb{H}$ be the Hilbert space associated with a physical system, referred to as the *object system*. The *properties* of the system are linear operators on $\mathbb{H}$. Its *state* is a statistical operator $\mathbb{W}$, i.e. a member of the set

$$\mathscr{S}=\{\mathbb{W} \mid \mathbb{W}=\mathbb{W}^{\dagger},\ \mathbb{W}>0, \operatorname{Tr}(\mathbb{W})=1\}.$$

We base our discussion on the standard principles of quantum theory and all results will be entirely consistent with it. Two alternative orthonormal bases in $\mathbb{H}$ are denoted $\{|\alpha\rangle \mid |\alpha\rangle \in \mathbb{H},\ \alpha \in \mathbb{I}_g\}$ and $\{|n\rangle \mid |n\rangle \in \mathbb{H},\ n \in \mathbb{I}_l\}$. The 'greek index set' $\mathbb{I}_g$ and the 'latin index set' $\mathbb{I}_l$ have the same cardinality (and are countable if and only if $\mathbb{H}$ is separable). The trace can be calculated in any basis; for instance

$$\operatorname{Tr}(\mathbb{W})=\sum_{\alpha \in \mathbb{I}_g}\langle\alpha|\mathbb{W}|\alpha\rangle=\sum_{n \in \mathbb{I}_l}\langle n|\mathbb{W}|n\rangle=1$$

with the usual notation for the scalar product in $\mathbb{H}$, etc.

The nature of the operators on $\mathbb{H}$, including the properties of the system, is defined in more detail in definition 2.5 later in the present section. The state $\mathbb{W} \in \mathscr{S}$ is a property of the system. The physical meaning of this standard postulate is recalled in more detail in the remark to proposition 3.1(i). Let $\mathscr{B}(\mathbb{H})$ be the set of bounded operators on $\mathbb{H}$, and let $\mathbb{C}$ be the complex number field. The state $\mathbb{W}$ defines a linear functional $\langle\cdot\rangle$: $\mathscr{B}(\mathbb{H}) \curvearrowright \mathbb{C}$: $\langle\mathbb{A}\rangle=\operatorname{Tr}(\mathbb{W}\mathbb{A})$ for $\mathbb{A} \in \mathscr{B}(\mathbb{H})$ and $\mathbb{W} \in \mathscr{S}$, called the *mean value of* $\mathbb{A}$ in the state $\mathbb{W}$. Even if $\mathbb{H}$ is not separable, we do not consider more general mean value functionals (sometimes called 'states').

Definition 2.2 (principle of objective causality).

(*a*) An *evolution* $\mathscr{E}$ is an ordered pair of states: $\mathbb{W}_{\text{old}} \curvearrowright \mathbb{W}_{\text{new}}$, where $\mathbb{W}_{\text{old}}, \mathbb{W}_{\text{new}} \in \mathscr{S}$.

(*b*) A *cause* is a set of bounded linear operators on $\mathbb{H}$

$$\mathscr{C}=\{\mathbb{E}_j \mid \mathbb{E}_j \in \mathscr{B}(\mathbb{H}), j \in \mathbb{J}\}.$$

(*c*) A *transformation* T is a positive map, defined by a cause $\mathscr{C}$, creating evolutions

$$\mathbb{W}_{\text{old}} \curvearrowright \mathbb{W}_{\text{new}}=T(\mathbb{W}_{\text{old}}) \equiv \sum_j \mathbb{E}_j \mathbb{W}_{\text{old}} \mathbb{E}_j^{\dagger}. \tag{1}$$

(*d*) The *graph* $\Gamma(T)$ of a transformation T is the set of all evolutions: $\mathbb{W}_{\text{old}} \curvearrowright \mathbb{W}_{\text{new}}=T(\mathbb{W}_{\text{old}})$ which may be caused by its cause $\mathscr{C}$.

(*e*) Two transformations T_1 and T_2 are *equivalent* when their graphs are identical: $\Gamma(T_1)=\Gamma(T_2)$.

Remarks

(*a*) Any change of state of the object system is an evolution. Generally an evolution takes time before it is completed.

(*b*) The elements in $\mathscr{C}$ are operators *of the object system itself* and they represent the causes of its change of state; hence 'objective causality'†. A displacement, for instance, may be caused by $\mathscr{C}=\{\mathbb{U}\}$, where $\mathbb{U}=\exp(-\mathrm{i}Q\mathbb{p}/\hbar)$. The object system operator $\mathbb{p}$, its property momentum, defines the *qualitative nature* of the transformation: displacement of the object. The parameter Q determines the *quantitative amount* of displacement and is a number which may refer to external systems (not parts of the object system); for instance, the meter sticks which define the (space) coordinates in terms of which Q is defined. Note that Q is *not* a property of the object system. The object system may have the property 'position', but Q is not it. The quantitative ingredients in the cause $\mathscr{C}$ can all be specified simultaneously. They represent those conditions which can be adequately described in terms of classical rational mechanics, classical electrodynamics and thermodynamics, say.

(*c*) For instance, if $\mathscr{C}=\{\mathbb{U}\}$, where $\mathbb{U}$ is unitary: $\mathbb{U}^\dagger=\mathbb{U}^{-1}$, then $\mathbb{W}_{\text{new}}=\mathbb{U}\mathbb{W}_{\text{old}}\mathbb{U}^\dagger$ is a *unitary* transformation. There are many other ways to change the state of a system, corresponding to transformations which are not unitary.

(*d*) The domain of T, and its range, are convex subsets of $\mathscr{S}$ (dom(T) consists of the states $\mathbb{W}\in\mathscr{S}$ whose images $T(\mathbb{W})$ belong to $\mathscr{S}$ and ran(T) consists of all the images, $T(\mathbb{W})$, of $\mathbb{W}\in$ dom(T)). When the domain of T coincides with $\mathscr{S}$, dom(T) $=\mathscr{S}$, then T is said to be *conventional* and its cause satisfies $\mathbb{F}\equiv\Sigma_j\,\mathbb{E}_j^\dagger\mathbb{E}_j=\mathbb{1}$. Thus the conventional transformations admit any state in $\mathscr{S}$ as initial state $\mathbb{W}_{\text{old}}$, while their ranges may, or may not, be restricted to subsets of $\mathscr{S}$.

Theorem 2.3 (objective causality). Every evolution $\mathscr{E}:\mathbb{W}_{\text{old}}\curvearrowright\mathbb{W}_{\text{new}}$ can be assigned at least one cause $\mathscr{C}$, such that $\mathbb{W}_{\text{new}}=T(\mathbb{W}_{\text{old}})$.

Proof. Given in Larsen (1986a). Allows $\mathscr{C}$ to be a countable set: $\mathbb{J}\subseteq\mathbb{Z}$.

Remark. This result allows for an extension of the scope of quantum theory—an extension which implements objective causality as a principle. Traditionally the scope has been restricted to unitary transformations, amended with the so-called projection postulate in order to define the physical interpretation of the theory (cf § 3). The objective causality formulated in (1) is a natural extension—physically as well as mathematically.

Example 2.4 (operations). Suppose that one temporarily expands the objective of the quantum theoretical description, so as to include an external system (some apparatus, for instance) together with the object system. Let the initial state of this combined system be $\mathbb{W}_{\text{old}}^{(\text{total})}=\mathbb{W}_{\text{old}}\otimes\mathbb{W}_{\text{old}}^{(\text{app})}$, defined on the Hilbert space $\mathbb{H}\otimes\mathbb{H}^{(\text{app})}$. Let $\mathbb{U}^{(\text{total})}$ be a unitary operator for the system of object and apparatus. Then a unitary transformation produces the evolution

$$\mathbb{W}_{\text{new}}^{(\text{total})}=\mathbb{U}^{(\text{total})}(\mathbb{W}_{\text{old}}\otimes\mathbb{W}_{\text{old}}^{(\text{app})})\mathbb{U}^{(\text{total})\dagger}.$$

† We use the concepts 'object', 'objective', 'cause' and 'causality' in the strict sense in which they were originally conceived. It is noteworthy that this is indeed possible in a contemporary operational context. We emphasise that some connotations which have become associated with these terms are *not* implied by the present terminology. Thus by 'objectivity' we mean no more, and no less, than what is expressed in the present text. 'Causality' means that every effect associated with the transformation of a system can be ascribed to a cause $\mathscr{C}$; and that a specification of the operational content of $\mathscr{C}$ suffices to 'explain' the reasons for these effects (cf Larsen 1986a). In particular, 'causality' does *not* imply 'classical determinism'.

The standard procedure is then to reduce to the object space $\mathbb{H}$, whereby

$$\mathbb{W}_{\text{new}} = \text{Tr}_{\text{app}}(\mathbb{W}^{(\text{total})}_{\text{new}}) \in \mathcal{S}$$

is a state of the object.

If we assume that $\mathbb{W}_{\text{new}}$ is the final state, then $\mathbb{W}_{\text{old}} \curvearrowright \mathbb{W}_{\text{new}}$ is an evolution for the object, known as an *operation* (Davies 1976, Kraus 1971, Lindblad 1976). It has been established that any operation can be represented in the form (1), so that an operation is a transformation with an objective cause. The most recent proof (Larsen 1986a) also shows that any conventional transformation can be expressed as an operation.

The formulation given above is not strictly objective, insofar as $\mathbb{U}^{(\text{total})}$ is not an operator pertaining to the object system exclusively. Also it is not strictly causal, insofar as the mathematical reduction by means of $\text{Tr}_{\text{app}}(\cdot)$ cannot be seen as a physical modification of the object system. However, there always exists an objectively causal transformation in terms of which one may 'explain what really happens to the object', regarded as an object physically operated upon.

For example, consider $\mathbb{U}^{(\text{total})} = \exp(-\text{i}\mathbb{Q}^{(\text{app})} \otimes \mathbb{p}/\hbar)$, where $\mathbb{Q}^{(\text{app})}$ is some property of the apparatus, defined as an operator on $\mathbb{H}^{(\text{app})}$ and $\mathbb{p}$ is an object property. Let $\{|j\rangle^{(\text{app})}\}$ be an orthonormal basis in $\mathbb{H}^{(\text{app})}$, consisting of eigenvectors of $\mathbb{Q}^{(\text{app})}$. Then $\mathbb{Q}^{(\text{app})}|j\rangle^{(\text{app})} = Q_j|j\rangle^{(\text{app})}$, where $\{Q_j\}$ are the eigenvalues of $\mathbb{Q}^{(\text{app})}$. Also, define $\xi_j = {}^{(\text{app})}\langle j|\mathbb{W}^{(\text{app})}_{\text{old}}|j\rangle^{(\text{app})}$, so that $\Sigma_j\, \xi_j = \text{Tr}_{\text{app}}(\mathbb{W}^{(\text{app})}_{\text{old}}) = 1$. Forming $\text{Tr}_{\text{app}}(\cdot)$ one then finds

$$\mathbb{W}_{\text{new}} = \sum_j \xi_j \mathbb{U}_j \mathbb{W}_{\text{old}} \mathbb{U}_j^\dagger$$

where $\mathbb{U}_j = \exp(-\text{i}Q_j\mathbb{p}/\hbar)$. The objective cause of this operation is therefore given by $\mathbb{E}_j = \alpha_j\mathbb{U}_j$, where $|\alpha_j|^2 = \xi_j$. Such a transformation we call *sub-unitary* (Larsen 1986a). Both the weights $\{\xi_j\}$, and the displacements $\{Q_j\}$, are determined externally by the nature and the state of the apparatus.

In the same way one can concentrate attention on the apparatus, regarded in itself as another object. Let the eigenvalues of the momentum $\mathbb{p}$ be $\{p_n\,|\,n \in \mathbb{I}_1\}$, defining the 'latin' basis on $\mathbb{H}$. The standard interpretation says that $P(n) = \langle n|\mathbb{W}_{\text{old}}|n\rangle$ is the predicted probability that the momentum attains the value p_n in the initial state $\mathbb{W}_{\text{old}}$ of the original object system. Thus one finds

$$\mathbb{W}^{(\text{app})}_{\text{new}} \equiv \text{Tr}(\mathbb{W}^{(\text{total})}_{\text{new}}) = \sum_n P(n)\mathbb{U}^{(\text{app})}_n \mathbb{W}^{(\text{app})}_{\text{old}} \mathbb{U}^{(\text{app})\dagger}_n$$

where $\mathbb{U}^{(\text{app})}_n = \exp(-\text{i}\mathbb{Q}^{(\text{app})}p_n/\hbar)$. The apparatus therefore also undergoes a sub-unitary transformation. Since the generator is the apparatus property $\mathbb{Q}^{(\text{app})}$, the transformation caused by $\mathbb{U}^{(\text{app})}_n$ might be called a 'boost' by an amount of momentum equal to p_n. The weights $\{P(n)\}$ and boosts $\{p_n\}$ are again determined from the outside, this time by the nature and state of the original object.

The interaction of the two systems, caused by this $\mathbb{U}^{(\text{total})}$, therefore tends to make them reciprocally record certain aspects of each other's initial states: the displacements are weighted by $\{\xi_j\}$ and the boosts by $\{P(n)\}$. Whether or not these evolutions qualify as measurements is a question which needs further investigation.

Both the object system and the apparatus system, on their own, end up in states $\mathbb{W}_{\text{new}}$ and $\mathbb{W}^{(\text{app})}_{\text{new}}$, respectively, with increased entropy (decreased purity (Larsen 1986a)). $\mathbb{U}^{(\text{total})}$ does not change the entropy of the total system of object and apparatus, so the apparent deficiency becomes invested in correlations which can only be detected if the two systems are regarded as a whole.

Definition 2.5 (aspect and idea). Let $\mathscr{L}_g = \{\mathbb{P}_\alpha \,|\, \mathbb{P}_\alpha = |\alpha\rangle\langle\alpha|, \alpha \in \mathbb{I}_g\}$ be the *complete* set of orthonormal projectors corresponding to an arbitrary orthonormal basis $\{|\alpha\rangle |\, |\alpha\rangle \in \mathbb{H}, \alpha \in \mathbb{I}_g\}$, i.e. $\mathbb{P}_\alpha \mathbb{P}_\beta = \delta_{\alpha\beta}\mathbb{P}_\alpha$ for all $\alpha, \beta \in \mathbb{I}_g$†.

(*a*) Let $\mathscr{A}_g$ denote the set of all compatible normal operators on $\mathbb{H}$ for which $\mathscr{L}_g$ provides simultaneous spectral decompositions: $\mathbb{A} = \Sigma_\alpha A_\alpha \mathbb{P}_\alpha$, for all $\mathbb{A} \in \mathscr{A}_g$, where $\mathrm{sp}(\mathbb{A}) = \{A_\alpha \,|\, \alpha \in \mathbb{I}_g\}$ is the spectrum of $\mathbb{A}$. We call $\mathscr{A}_g$ an *aspect* of the system‡.

(*b*) The self-adjoint members of $\mathscr{A}_g$ are referred to as *properties*, for which $\mathrm{sp}(\mathbb{A}) \subseteq \mathbb{R}$.

(*c*) A subset $\mathscr{I}$ of properties in $\mathscr{A}_g$ is called an *idea*; and when an idea is sufficiently rich that its spectrum may label the basis $\{|\alpha\rangle\}$ it is referred to as a *complete idea*, $\mathscr{I}_c$ (i.e. a 'complete set of commuting operators') and $\mathscr{I}, \mathscr{I}_c \subseteq \mathscr{A}_g$.

(*d*) The aspect $\mathscr{A}_g$ includes at least one complete idea, namely $\mathscr{L}_g$, referred to as the (complete) *logical idea* of the aspect because its projectors represent the ideas of propositions in classical logic.

Remark. In general the nature of a system includes aspects which are not compatible, i.e. whose elements do not all commute and do not all have spectral decompositions on a common logical idea. Such incompatible ideas define the potentiality of performing incompatible operations on the system (implying complementarity). Thus, if $\mathscr{A}_g$ and $\mathscr{A}_l$ are incompatible aspects they contain at least one incompatible pair: $\mathbb{B} \in \mathscr{A}_g$, $\mathbb{A} \in \mathscr{A}_l$ (for which $[\mathbb{B}, \mathbb{A}] \neq 0$, assuming the commutator is well defined (cf Thirring 1981)). The alternative aspect $\mathscr{A}_l$ is defined by the logical idea $\mathscr{L}_l = \{\mathbb{P}_n \,|\, \mathbb{P}_n = |n\rangle\langle n|, n \in \mathbb{I}_l\}$, where $\{|n\rangle\}$ is an alternative orthonormal basis in $\mathbb{H}$, indicated by quantum numbers $n \in \mathbb{I}_l$.

Example 2.6. The Pauli operators σ^x, σ^y and σ^z on a two-dimensional Hilbert space define properties of the 'binary system' (spin $\frac{1}{2}$). They belong to incompatible (and complementary) aspects. Each component $\sigma^k = \hat{k} \cdot \vec{\sigma}$, where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ and $\hat{k}$ is a unit vector, defines an aspect of the binary system and represents the (complete) idea of rotation about $\hat{k}$. The logical idea in the $\hat{k}$ aspect consists of two orthogonal projectors: $\mathbb{P}_- = (\mathbb{1} - \sigma^k)/2$ and $\mathbb{P}_+ = (\mathbb{1} + \sigma^k)/2$, i.e. $\mathscr{L} = \{\mathbb{P}_-, \mathbb{P}_+\}$. The set of all aspects of the binary system is isomorphic to the points on the unit hemisphere. Hence there are uncountably many incompatible aspects of even this, the simplest of all systems.

3. Axioms

Consider an arbitrary object state $\mathbb{W}$, with respect to which the arbitrary aspect $\mathscr{A}_l$ is to be measured. The following three criteria (i)–(iii) define the class of conventional transformations which serve as 'complete measurements' of this aspect.

† Here $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, and $\delta_{\alpha\beta} = 0$ otherwise.

‡ Technical details pertaining to these definitions are discussed in the appendix.

Proposition 3.1. $\mathcal{M}(\mathscr{A}_l)$ is the class of *complete measurements* of $\mathscr{A}_l$, provided for all $T \in \mathcal{M}(\mathscr{A}_l)$ and all pure states (projectors) $\mathbb{P}_n \in \mathscr{L}_l \subset \mathscr{A}_l$:

(i)	$[\mathbb{P}_n, T(\mathbb{W})] = 0$	(compatibility)
(ii)	$T(\mathbb{P}_n) = \mathbb{P}_n$	(reproducibility)
(iii)	All T are equivalent	(objectivity).

Remarks

(i) *Compatibility.* The measurement transformation $T \in \mathcal{M}(\mathscr{A}_l)$ creates a new state $T(\mathbb{W}) \in \mathscr{A}_l$. This means that (necessarily†)

$$T(\mathbb{W}) = \mathbb{w} = \sum_n w_n \mathbb{P}_n \tag{2}$$

where $\{w_n\} = \mathrm{sp}(\mathbb{w})$ are statistical weights in the situation after the measurement is performed. In general $\mathbb{W}$ and $\mathbb{w}$ belong to different (incompatible) aspects, i.e. $\mathbb{W} \in \mathscr{A}_g$ and $\mathbb{w} \in \mathscr{A}_l$. The initial statistical weights $\{W_\alpha\} = \mathrm{sp}(\mathbb{W})$, corresponding to $\mathbb{W} = \Sigma_\alpha W_\alpha \mathbb{P}_\alpha$, pertain to a situation, qualitatively defined by $\mathscr{A}_g$ and quantitatively by $\{W_\alpha\}$, which is not compatible with the one that is required in order for the measurement to be an empirical fact.

The reason is this. As is well known (see von Neumann 1955, ch 5) a system can be separated (without interfering with its constitution) into subsystems in pure states, belonging to the logical idea of the aspect to which its state belongs, but *only* into these‡. Thus the measurement enables us to render $\mathbb{w}$ *empirically* equivalent to an ensemble of subsystems in pure states $\mathbb{P}_n \in \mathscr{A}_l$, with relative weights $\{w_n\}$. Here $w_n = N_n/N$, where N_n is the number of specimens in the nth subsystem and $N = e\Sigma_n N_n$ = 'total number of specimens in the original system'†. If $\mathbb{A} \in \mathscr{A}_l$ is a property belonging to the measured aspect, the weights $\{w_n\}$ represent the *experimental statistic* of its quantitative values, taken from $\{A_n\} = \mathrm{sp}(\mathbb{A})$, i.e. 'data'.

On the other hand, properties belonging to aspects incompatible with $\mathscr{A}_l$, for example $\mathbb{B} \in \mathscr{A}_g$, cannot simultaneously with $\mathbb{A}$ be assigned quantitative values. In particular, the original assignments, say values from $\{B_\alpha\} = \mathrm{sp}(\mathbb{B})$ with weights $\{W_\alpha\}$ in the initial state $\mathbb{W}$, are destroyed by the measurement transformation. Hence the

† The statistical operators $\mathbb{W} \in \mathscr{S}$ are trace-class, and hence compact. Therefore their spectra, such as $\mathrm{sp}(\mathbb{W}) = \{W_\alpha\}$ and $\mathrm{sp}(\mathbb{w}) = \{w_n\}$, are discrete (countable), consist of eigenvalues and their only point of accumulation is 0. This is true whether or not $\mathbb{H}$ is separable (Hilbert–Riesz theorem (von Neumann 1955, Thirring 1981, Dunford and Schwartz 1963, Conway 1985)). Therefore it is always feasible to separate $\mathbb{w}$, say, into a countable set of subsystems in orthogonal pure states (as required for reasons of physics). In particular, every member of $\mathscr{S}$ can be diagonalised as in (2) (cf the appendix for further remarks).

‡ The decomposition of a given state $\mathbb{W} \in \mathscr{S}$ into a convex combination of pure states (i.e. $\mathbb{W} = \Sigma_j \xi_j \mathbb{P}_j$) is not mathematically unique, unless the pure states are required to belong to a set of orthogonal projectors (belonging to an aspect) (Schrödinger 1936). It is well known that mixed states formed by the convex combination of non-orthogonal pure states cannot be separated without interfering with the system, such that when recombined the state becomes other than the original $\mathbb{W}$ (namely one of higher entropy (von Neumann 1955)). Separation and reassembly is feasible if and only if the components are orthogonal, say pure subsystem states (i.e. $\mathbb{W} = \Sigma_\alpha W_\alpha \mathbb{P}_\alpha$, where $\{\mathbb{P}_\alpha\} \subseteq \mathscr{L}_g$ are orthogonal).

measurement changes the concrete character of the system from one belonging to the 'greek' aspect $\mathscr{A}_g$ into one belonging to the 'latin' aspect $\mathscr{A}_l$ which is measured†.

(ii) *Reproducibility.* If the initial state is compatible with the measured aspect (i.e. $\mathscr{A}_g = \mathscr{A}_l$) the measurement must confirm it. It suffices to require that T reproduces any of the pure states $\mathbb{W} = \mathbb{P}_n$.

(iii) *Objectivity.* The result $\mathbb{w}$ will, of course, depend on the object state $\mathbb{W}$. Apart from that, it must depend only on the measured aspect $\mathscr{A}_l$ in the way that is implied by (i) and (ii). 'Objectivity' demands that a measurement mapping $\mathbb{W} \rightarrow \mathbb{w}$ must depend in no other way on $\mathscr{C}$. If there are different ways to perform the measurement its *result* $\mathbb{w}$ must be independent of whatever distinguishes such equivalent transformations. It must not depend on which apparatus is used, not even on differences between alternative procedures that may work with the same equipment. Otherwise $\mathbb{w}$ will not be a faithful 'picture' of $\mathbb{W}$ (the problem of 'systematic errors'). If the experimenter knows how the external systems take part in determining the cause $\mathscr{C}$ he can still tell the difference between transformations, even though they are equivalent with respect to the states of the object (mathematically all $T \in \mathscr{M}(\mathscr{A}_l)$ have the same graph).

The reason that these criteria can be regarded as axiomatic is that, in essence, they express the empirical distinction between alternatives in a classical logic, symbolised in the logical idea $\mathscr{L}_l$ of the aspect to be measured. This is the most elementary axiom of a rational empirical theory‡. It makes no difference that this axiom is embedded in an operational context, which constitutes the 'quantum postulate'. Remarkably, however, as will be shown in the following, measurements can thus be defined without assuming anything which is not already necessary to define the *empirical* concept of 'state', i.e. the 'statistical interpretation' of the entity $\mathbb{w}$, which relies on the classical logic of $\mathscr{L}_l$. In proposition 3.1 nothing whatsoever is postulated regarding *how* the measurement transformation takes place.

In contrast, when quantum theory is based on the traditional projection postulate a measurement is *defined* as the particular transformation caused by $\mathscr{C} = \mathscr{L}_l = \{\mathbb{P}_n\}$, the 'logical cause'. According to this postulate the measurement is a definite transformation

$$\mathbb{w} = \sum_n \mathbb{P}_n \mathbb{W} \mathbb{P}_n \tag{3}$$

the effect of which is the projection of $\mathbb{W}$. Even though there can be no doubt—and our results confirm this—that the projection postulate is empirically justified, it is virtually impossible to motivate it in an axiomatically simple way. It has always rested entirely on its considerable merit. Rather, we shall view (3) as a statement of accomplished fact: that a transformation has taken place, starting with $\mathbb{W}$ and resulting in $\mathbb{w}$, a transformation which evidently agrees with the axiomatic critieria (i)-(iii). But

† This is how the transformations in $\mathscr{M}(\mathscr{A}_l)$ produce the desired measurement result. In § 5 we give examples of how to record data in an external apparatus (see also example 2.4). But it is well known how to do this by establishing correlations (von Neumann 1955, Herbut and Vujičić 1976, Zurek 1982, Walls *et al* 1985). The decisive step is the transformation into the object state $\mathbb{w}$, given by (2) (Wigner 1983).

‡ It is well known that rational theories can be based on non-classical logic. The emphasis is on 'empirical', on the axiom that *actual fact* (i.e. 'physical reality') does conform to classical logic. The probabilistic structure of quantum theory is not reducible to classical logic. But these probabilities are *theoretical predictions* for which no such assumption seems axiomatic, as long as the predictions are not all supposed to apply to the *same* actual situation (cf § 6). On the other hand, the states $\mathbb{W}$ and $\mathbb{w}$ of quantum theory always represent what is *actually* the case, and therefore incorporate a statistic, $\{W_\alpha\}$ and $\{w_n\}$, respectively, based on the classical logic of their aspects.

(3) offers no physical explanation of *how* it took place. As the cause $\mathscr{C} = \mathscr{L}_1$ shows, the projection is a 'logical' rather than a 'physical' transformation.

4. Central theorem

Theorem 4.1. Let the transformation $T \in \mathcal{M}(\mathscr{A}_1)$ be an arbitrary complete measurement of the aspect $\mathscr{A}_1$, and let its cause $\mathscr{C}$ consist of operators $\mathbb{E}_j = \alpha_j \Psi_j$, where $\|\Psi_j\| = 1$ and $|\alpha_j|^2 = \xi_j = |\mathbb{E}_j^\dagger \mathbb{E}_j\|$. Then we find the following.

(*a*) The cause belongs to the measured aspect: $\mathscr{C} \subseteq \mathscr{A}_1$.

(*b*) The cause satisfies the *measurement condition*

$$\sum_j \xi_j \Psi_k^{(j)} \Psi_n^{(j)*} = \delta_{kn} \qquad \text{for } k, n \in \mathbb{I}_1 \tag{4}$$

where for all j: $\mathrm{sp}(\Psi_j) = \{\Psi_n^{(j)} \mid n \in \mathbb{I}_1\}$.

(*c*) The *measured statistic* is

$$w_n = \langle n | \mathbb{W} | n \rangle \equiv P(n) \tag{5}$$

where $\{P(n) \mid n \in \mathbb{I}_1\}$ must therefore be the *predicted probabilities* to be assigned *theoretically* to the aspect $\mathscr{A}_1$ in the initial object state $\mathbb{W}$.

(*d*) Every result is reproducible by a repeated measurement:

$$T(\mathbb{w}) = \mathbb{w}. \tag{6}$$

Proof. According to the compatibility criterion (i) we must have

$$\langle k | \mathbb{w} | n \rangle = w_n \delta_{kn} = \sum_j \sum_{lm} \langle k | \mathbb{E}_j | l \rangle \langle l | \mathbb{W} | m \rangle \langle m | \mathbb{E}_j^\dagger | n \rangle \tag{7}$$

where all latin quantum numbers belong to $\mathbb{I}_1$. The reproducibility criterion (ii) requires $w_n = \delta_{nn'}$ for $\mathbb{W} = \mathbb{P}_{n'}$ and any $n' \in \mathbb{I}_1$. Thus

$$\delta_{kn}\delta_{nn'} = \sum_j \xi_j \langle k | \Psi_j | n' \rangle \langle n' | \Psi_j^\dagger | n \rangle \tag{8}$$

and, in particular, for $k = n$:

$$\sum_j \xi_j |\langle n | \Psi_j | n' \rangle|^2 = \delta_{nn'}.$$

Since the terms in this sum are non-negative they must all vanish for $n \neq n'$. Thus, for all n, $n' \in \mathbb{I}_1$:

$$\langle n | \Psi_j | n' \rangle = \Psi_n^{(j)} \delta_{nn'}. \tag{9}$$

This provides a spectral decomposition of each Ψ_j on $\mathscr{L}_1$, i.e. $\Psi_j = \Sigma_n \Psi_n^{(j)} \mathbb{P}_n$, where $|\Psi_n^{(j)}| \leq \|\Psi_j\| = 1$, and proves (*a*). From (7) then

$$w_n \delta_{kn} = \langle k | \mathbb{W} | n \rangle \sum_j \xi_j \Psi_k^{(j)} \Psi_n^{(j)*}.$$

Since the measurements in $\mathcal{M}(\mathcal{A}_1)$ admit arbitrary $\mathbb{W}$ this δ_{kn} must come from the second factor, which establishes (b) and (c)†, and (d) follows immediately from (c) or directly from proposition 3.1(ii) using (2).

Remarks

(a) The measurement provides the empirical basis for assigning quantitative values to the physical properties belonging to the measured aspect. It does this in a way which faithfully (i.e. conforming to the criterion (iii) of objectivity) reflects the initial object state. In order to achieve this actualisation of the chosen aspect one must perform a transformation, on the object system, *caused by the very operators which represent the measured properties.* Therefore the operators of quantum theory *qualitatively* define the very operations which generate their own *quantitative* realisation. The principle of 'objective causality' thus provides a most explicit motivation for the 'quantum postulate', according to which the properties of systems are operationally defined entities (definition 2.1).

(b) The cause $\mathscr{C}$ of the measurement consists of a *definite* selection of operators from the measured aspect. *The measurement transformation therefore consists of a definite concert of operations.* Such experimental design (cf example 4.3) requires a careful tuning of the parameters in the elements of $\mathscr{C}$, parameters which are determined by external systems (apparatus). Without such tuning one shall not achieve a (complete) measurement, but merely have performed some transformation on the system. Of course, transformations take place spontaneously all the time; but in order to have a measurement a deliberate design by an experimenter is (generally) necessary.

On the other hand, the measurement condition (4) is not all that hard to satisfy. It is essentially a condition of orthonormality imposed on the quantities $\Psi_n^{(j)}$, as enumerated by n, and with variable j and weight function ξ_j. This latitude in the design conditions is the reason that the measurement class $\mathcal{M}(\mathcal{A}_1)$ includes an infinity of equivalent transformations. The 'projection' (3), for example, has $\Psi_n^{(j)} = \delta_{jn}$, but there are many alternatives.

(c) This result establishes the empirical foundation of the quantum theoretic probability calculus. One does not have to postulate that $P(n) = \langle n|\mathbb{W}|n\rangle$, or even that $P(n) = |\langle n|\psi\rangle|^2$ for a pure state $\mathbb{W} = |\psi\rangle\langle\psi|$, is a probability and to support it empirically with the 'projection postulate'. The theorem shows that *any* faithful 'picture' of the original object state $\mathbb{W}$, taking the empirical form of a reproducible actualisation of one of the aspects of the system, must necessarily produce a statistic $\{w_n\}$ which agrees with a particular theoretical construction, namely the quantities $\{\langle n|\mathbb{W}|n\rangle\}$, which *must therefore be the predicted probabilities* $\{P(n)\}$.

Of course, on the basis of the state $\mathbb{W}$ the theorist is free to predict probabilities for every aspect of the system, even when the aspects are not compatible. However,

† According to the classification of transformations/evolutions proposed in Larsen (1986a), all (complete) measurements $T \in \mathcal{M}(\mathcal{A}_1)$ are 'conventional', i.e. admit arbitrary initial object states ($\mathrm{Tr}(T(\mathbb{W})) = 1$ for all $\mathbb{W} \in \mathcal{S}$). Under this assumption it turns out that (i) and (ii) alone imply that there is only one equivalence class $\mathcal{M}(\mathcal{A}_1)$ (definition 2.2(e)) and the outcome, $w = T(\mathbb{W})$, turns out not to depend on the cause $\mathscr{C}$ except to the extent that $\mathscr{C}$ secures that w belongs to the measured aspect $\mathcal{A}_1$. Our earlier proof (Larsen 1986b) did depend on (iii) and the present theorem is also stronger in other respects. There are circumstances in which the objectivity (iii) becomes relevant as an axiom. One is in incomplete measurements (cf a subsequent paper), where it turns out that there are several equivalence classes (because less is required of an incomplete measurement). Another would occur if one were to consider 'unconventional' transformations ($\mathrm{Tr}(\mathbb{W}) \neq 1$ for some $\mathbb{W} \in \mathcal{S}$ outside the domain of T). However, to generalise in this way, although interesting, is beyond the scope of the present investigation.

in order to have the predictions experimentally tested, which means seeking evidence that the theoretically assumed $\mathbb{W}$ is really the state of a given object system, a definite one of these aspects must be singled out for measurement. It is a free choice, but between mutually exclusive alternatives. In fact, part (*a*) of the theorem shows that measurements of incompatible aspects *must* correspond to incompatible operations, because the causes of the transformations which constitute the measurements of incompatible aspects are themselves incompatible.

On the principle of 'objective causality' the present theorem therefore tells us both *how* to perform measurements in practice and *how* to make theoretical predictions about their outcomes. It also tells us that there is no way in which incompatible aspects can be measured simultaneously. None of these fundamentals of quantum theory therefore *have to* be based on postulate, as they have been traditionally.

(*d*) The present theorem encompasses every transformation in which the quantitative actualisation of a physical aspect (by (i)) is reproducible (by (ii)). Objectivity (iii) is actually a corollary†.

Corollary 4.2. In the state $\mathbb{W}$ the theoretically predicted *mean* $\langle \mathbb{A} \rangle$ which agrees with the *average* $\bar{A}$ over the experimental statistic $\{w_n\}$, produced in a (complete) measurement of the aspect $\mathscr{A}_1$ to which a bounded $\mathbb{A}$ belongs, must be given by

$$\bar{A} \equiv \sum_n A_n w_n = \mathrm{Tr}(\mathbb{W}\mathbb{A}) \equiv \langle \mathbb{A} \rangle. \tag{10}$$

Proof. By (5)

$$\bar{A} = \sum_n A_n \langle n|\mathbb{W}|n\rangle = \sum_n \langle n|\mathbb{W}\mathbb{A}|n\rangle = \mathrm{Tr}(\mathbb{W}\mathbb{A})$$

since the trace $\mathrm{Tr}(\cdot)$ can be evaluated in any basis in $\mathbb{H}$ and $\mathbb{W}\mathbb{A}$ belongs to the operator trace class when $\mathbb{A} \in \mathscr{B}(\mathbb{H})$ (Thirring 1981, Dunford and Schwartz 1963).

Remark. Let $\mathbb{A} = \mathbb{A}_1 + \mathbb{A}_2$, where $\mathbb{A}$, $\mathbb{A}_1$ and $\mathbb{A}_2$ are bounded operators, possibly incompatible, i.e.belonging to different aspects. Then

$$\langle \mathbb{A} \rangle = \langle \mathbb{A}_1 \rangle + \langle \mathbb{A}_2 \rangle. \tag{11}$$

This result, the linearity of the quantum theoretic mean-value functional, is also a matter related to the interpretation of the theory and is postulated in the traditional version (von Neumann 1955). It could be a source of concern, representing a consequence of the 'projection postulate' which cannot be directly tested because the three mean values refer to mutually exclusive experimental situations, i.e. require measurements of incompatible aspects. The corollary shows that no measurements, by 'projections' or otherwise, allow a violation of (11).

Example 4.3. Let $\mathbb{A} \in \mathscr{A}_1$ be the property defined by $\mathrm{sp}(\mathbb{A}) = \{A_n \mid A_n = a_0 + an,\ n \in \mathbb{I}_1\}$ and let it be the generator of the set $\{\Psi_j\}$ of unitary operators

$$\Psi_j = \exp(-\mathrm{i}Q_j\mathbb{A}/\hbar) \tag{12}$$

† See footnote on previous page.

where $\{Q_j\}$ are parameters designed by the experimenter and determined by means of external apparatus systems. Their spectra are given by $\Psi_n^{(j)} = \exp[-\mathrm{i}Q_j(a_0 + an)/\hbar]$. According to the measurement condition (4) the experimenter must therefore design the sets $\{\xi_j\}$ and $\{Q_j\}$ to satisfy

$$\sum_j \xi_j \exp[-\mathrm{i}Q_j a(k-n)/\hbar] = \delta_{kn}. \tag{13}$$

Let $\mathbb{H}$ be a finite space, $\dim(\mathbb{H}) = g$ and $\mathbb{I}_1 = \{0, 1, \ldots, g-1\}$. Then (13) is satisfied with a set of G operators $\{\Psi_j\}$ if, for instance, for $j = 0, 1, \ldots, G-1$:

$$\xi_j = 1/G \qquad \text{and} \qquad Q_j = Q_0 + jQ \tag{14}$$

provided

$$Q = h/aG \tag{15}$$

and with arbitrary a_0, Q_0, and for any $G \geqslant g$. Even within this rather special design there is an infinity of causally distinct experimental options. On the other hand, the measurement conditions (14) and (15) are quite specific.

In the binary system, $g = 2$, the aspect corresponding to the space direction $\hat{k}$ can be measured by means of a set of rotations about $\hat{k}$, whose unitary operators are generated by the spin $\mathbb{A} = \hbar\sigma^k/2$, with $a = \hbar$. The present design consists in an even selection of $G \geqslant 2$ angles $\{Q_j\}$ evenly spaced around the equatorial unit circle: $Q = 2\pi/G$. Both the number G and the overall orientation angle Q_0 are arbitrary.

5. Discussion: the binary system (spin $\frac{1}{2}$) as object

The measurements caused by 'sub-unitary' transformations, as in example 4.3, involve a set $\{\mathbb{W}_j\}$ of subsystem states

$$\mathbb{W}_j = \Psi_j \mathbb{W} \Psi_j^\dagger. \tag{16}$$

None of these, however, are individually compatible with the aspect $\mathscr{A}_1$ to be measured. To achieve the actualisation of $\mathscr{A}_1$, according to the compatibility criterion (i), it is *necessary* to establish the convex combination (i.e. $\Sigma_j\, \xi_j = 1$)

$$\mathbb{w} = \sum_j \xi_j \mathbb{W}_j. \tag{17}$$

Indeed, theorem 4.1 encompasses every conceivable way to accomplish this; hence the necessity of designing some appropriate convex combination according to the measurement condition (4). Even if the subsystems may happen to be distinguishable, their states $\{\mathbb{W}_j\}$ are by no means representative polls of the state $\mathbb{w}$ of the total system and they belong to aspects which are generally not compatible, neither with $\mathscr{A}_1$ nor with each other.

As the present new measurement transformations have 'physical causes', it is clear that they can be regular physical processes and take time to carry through.

(a) In the binary system it is particularly clear what happens during the measurement. The states of the binary system are uniquely defined by the polarisations $\mathbb{W} = \frac{1}{2}(\mathbb{1} + \vec{P} \cdot \vec{\sigma})$, where $\vec{P} = \mathrm{Tr}(\mathbb{W}\vec{\sigma})$ belongs to the unit ball in space, with the pure quantum states on its surface. After the measurement $\mathbb{w} = \frac{1}{2}(\mathbb{1}\vec{p} \cdot \vec{\sigma})$, where $\vec{p} = \mathrm{Tr}(\mathbb{w}\vec{\sigma})$.

A unitary transformation rotates the polarisation in space: $\mathbb{W}_j = \frac{1}{2}(\mathbb{1} + \vec{P}_j \cdot \vec{\sigma})$, where $\vec{P}_j$ is the rotated version of $\vec{P}$ which corresponds to Ψ_j. Thus

$$\vec{p} = \sum_j \xi_j \vec{P}_j. \tag{18}$$

When the $\hat{k}$ aspect has been measured it turns out that $\vec{p}$ is the geometric projection: $\vec{p} = (\hat{k} \cdot \vec{P})\hat{k}$. Evidently this purely geometric operation corresponds to the logical cause inherent in the projection postulate†.

A concert of rotations, such as defined in example 4.3, corresponds to a physical cause. The components of $\{\vec{P}_j\}$ transverse to $\hat{k}$ form a regular G-gon in the equatorial plane to $\hat{k}$. Hence $\vec{p}$ has no transverse component and the longitudinal component is again $\hat{k} \cdot \vec{P}$. In the limit $G \to \infty$ one obtains the familiar Stern-Gerlach method, in which the measurement of spin is caused by an inhomogeneous external magnetic field designed by the experimenter so that its effect is the required concert of rotations (further details are given below).

In fact, any polygon, regular or not, formed of rotated and weighted replicas of $\vec{P}$, i.e. the transverse components of the vectors $\{\xi_j \vec{P}_j\}$, represents a causally distinct measurement of the $\hat{k}$ aspect. The measurement condition requires that, in any such design, the transverse components close up to form a polygon. The class of equivalent measurement transformations is therefore uncountably infinite for any $\hat{k}$ aspect. Every one of its elements can handle an arbitrary initial object polarisation, whether it be known in advance of the measurement or not. We have thus established the following corollary to theorem 4.1.

Corollary 5.1. The class $\mathcal{M}(\mathcal{A})$, which defines the complete measurement of an arbitrary aspect $\mathcal{A}$, in general contains an uncountable infinity of causally distinct but equivalent transformations.

Proof. By demonstration, as in the example discussed above.

(*b*) Suppose we employ a second spin-$\frac{1}{2}$ system as apparatus, after the fashion of example 2.4 in § 2. Quite remarkably, one finds that such an operation can *both* perform the measurement on the object *and* record the result in the apparatus.

Let $\mathbb{U}^{(\text{total})} = \exp(-\mathrm{i}\mathbb{H}^{(\text{total})} t/\hbar)$, where t is the time elapsed since the beginning of the measurement and $\mathbb{H}^{(\text{total})} = J\sigma^{z(\text{app})} \otimes \sigma^k$ is a Hamiltonian designed to measure the $\hat{k}$ aspect of the object spin (J is some coupling constant of dimension energy). Since

† It is a somewhat peculiar feature of quantum theory—although not particularly a consequence of the present formulation of measurements—that the unpolarised state $\mathbb{w} = \frac{1}{2}\mathbb{1}$ (i.e. $\vec{p} = \vec{0}$) may be the result of measuring different aspects. The reason is that such a state as this belongs to different (incompatible) aspects ($\mathbb{1}$ commutes with all operators). For instance, if one measures σ^y or σ^z in a state $\mathbb{W} = \frac{1}{2}(\mathbb{1} + P^x \sigma^x)$, which has polarisation in the x direction, then $\mathbb{w} = \frac{1}{2}\mathbb{1}$ confirms the prediction that both $\sigma^y = \pm 1$ and $\sigma^z = \pm 1$ are equally probable in the state $\mathbb{W}$. Of course, this is what it means to say that $\mathbb{W}$ has no polarisation transverse to the x direction. The completely mixed state $\mathbb{w} = \frac{1}{2}\mathbb{1}$ is the state that is furthest away from all the pure states; in 3-space $\vec{P} = \vec{0}$ is the centre of the unit ball. Such a state $\mathbb{w} = (1/g)\mathbb{1}$ exists in all finite spaces $\dim(\mathbb{H}) = g$, is unique, and is equidistant from all pure states in the metric defined by the supremum norm $\|\cdot\|$. *Proof.* Let $\mathbb{A} = \mathbb{P}_\alpha - (1/g)\mathbb{1}$, where $\mathbb{P}_\alpha$ is any pure state, with $\mathrm{sp}(\mathbb{A}) = \{1 - 1/g, -1/g\} \equiv \{A_\alpha\}$. Then $\|\mathbb{A}\| = \|\mathbb{P}_\alpha - (1/g)\mathbb{1}\| = \sup_\alpha |A_\alpha| = 1 - 1/g = \mathrm{dist}(\mathbb{P}_\alpha, (1/g)\mathbb{1})$ independently of $\mathbb{P}_\alpha$. No state is further away from all the pure states, since for $\mathbb{W} \neq (1/g)\mathbb{1}$ one can always find a $\mathbb{P}_\alpha$ which is closer than $1 - 1/g$. For instance, take $\mathbb{P}_\alpha$ from the logical idea that defines the aspect of such a $\mathbb{W}$, with α corresponding to the largest eigenvalue W_α. Then the distance is $1 - W_\alpha < 1 - 1/g$ since $\max_\alpha(W_\alpha) > 1/g$.

this case corresponds to $G=2$ the measurement may first be accomplished at the time $t=\pi\hbar/4J$ (and periodically thereafter, period $=\pi\hbar/2J$) when $\mathbb{U}_j=\exp(-\mathrm{i}Jtj\sigma^k/\hbar)$ has rotated the object through either of the angles $\pm\pi/2$. Here $\mathbb{J}$ is taken to consist of the spectral values $j=\sigma^{z(\mathrm{app})}=\pm 1$ for the component $\sigma^{z(\mathrm{app})}$ of the apparatus spin $\vec{\sigma}^{(\mathrm{app})}$ along an arbitrary direction which is taken to be the z axis.

The initial apparatus state is $\mathbb{W}^{(\mathrm{app})}_{\mathrm{old}}=\frac{1}{2}(\mathbb{1}+\vec{P}^{(\mathrm{app})}\cdot\vec{\sigma}^{(\mathrm{app})})$. So in order to fulfil the design criteria one must arrange to have ${}^{(\mathrm{app})}\langle j|\mathbb{W}^{(\mathrm{app})}_{\mathrm{old}}|j\rangle^{(\mathrm{app})}=\xi_j=\frac{1}{2}$ for $j=\pm 1$. There is little design latitude with such a small apparatus, except that the choice of the direction called the z axis is arbitrary. This requires that $\vec{P}^{(\mathrm{app})}$ is in the xy plane and one can let it define the x axis: $\vec{P}^{(\mathrm{app})}=(P^{(\mathrm{app})},0,0)$, where $|P^{(\mathrm{app})}|\leqslant 1$ is the initial apparatus polarisation.

When calibrated in this way the apparatus performs the measurement as described above. To see the recording, reduce to the apparatus space $\mathbb{H}^{(\mathrm{app})}$. If $n=\sigma^k=\pm 1$, then the predicted probabilities are $P(n)=\frac{1}{2}[1+n(\hat{k}\cdot\vec{P})]$, where $\vec{P}$ is the initial object polarisation (and $w_n=P(n)$ is the statistic obtained). The final object polarisation is in the $\hat{k}$ direction and is $(\hat{k}\cdot\vec{P})=P\cos\theta_{kP}$, where $P=|\vec{P}|$ and θ_{kP} is the angle between $\vec{P}$ and $\hat{k}$.

The apparatus polarisation also gets rotated through either of the angles $\pm\pi/2$, but about the z axis. These two mixing alternatives are weighted as $P(n=\pm 1)$. The final apparatus polarisation then becomes directed along the y axis:

$$\vec{P}^{(\mathrm{app})}_{\mathrm{new}}=\left(0,\sum_{n=\pm 1}nP(n)P^{(\mathrm{app})},0\right)=(0,(\hat{k}\cdot\vec{P})P^{(\mathrm{app})},0).$$

The factor $(\hat{k}\cdot\vec{P})$ therefore records the projection of the initial object polarisation along the $\hat{k}$ direction, which is indeed the result of the measurement.

Note that both object and apparatus polarisations end up smaller than they were initially. This means that—on their own—both object and apparatus emerge in less pure states (the spin-$\frac{1}{2}$ purity is $\frac{1}{2}(1+P^2)$), but have become correlated.

It is somewhat surprising to discover how much can be achieved by such simple devices. By design the y aspect of the apparatus spin acts as a binary register: separate $\sigma^{y(\mathrm{app})}=\pm 1$ and counting reveals what $(\hat{k}\cdot\vec{P})$ is (since $\vec{P}^{(\mathrm{app})}$ is known). One could, of course, also separate according to the $\hat{k}$ aspect of the object spin and obtain the same information. There is no 'macroscopic' component of the present apparatus, although there *is* irreversibility—insofar as it might not be feasible to count in these reduced modes and to fully retrieve the object-apparatus correlation at the same time. The latter would require us to measure another aspect of the combined object and apparatus system than that to which $\sigma^{y(\mathrm{app})}\otimes\sigma^k$ belongs, namely that aspect to which its final state $\mathbb{W}^{(\mathrm{total})}_{\mathrm{new}}$ belongs†.

It therefore appears that a macroscopic device would only be required if one wanted to fix the outcome of the measurement into a time-independent state by some kind of irreversible relaxation in the course of time (Zurek 1982, Walls *et al* 1985). In the present design one is obliged to either turn off the object-apparatus interaction $\mathbb{H}^{(\mathrm{total})}$ at a precise instant, or to catch the spins precisely at the recurring instants when the measurement criterion is periodically satisfied. However, if one can do that, then the

† This is not feasible, for the following reason. Which aspect $\mathbb{W}^{(\mathrm{total})}_{\mathrm{new}}$ belongs to depends on the initial state $\mathbb{W}_{\mathrm{old}}$ of the object, among other things. The experimenter would therefore have to know the result of the measurement beforehand (in fact, he would have to know more, namely all of $\mathbb{W}_{\mathrm{old}}$). But, of course, the point of measuring is (primarily) to aquire information about the object which is not already available.

collection of apparatus spins, say, would seem to constitute as permanent a record as one could wish.

(c) We conclude by discussing an example which demonstrates how a coupling of the spin-$\frac{1}{2}$ object to a 'large' environment causes the object to tend to permanently maintain the measurement result in the course of time, with increasing accuracy the 'larger' the environment is.

For instance, one may use a spin S as apparatus, in a manner similar to the way in which a spin $\frac{1}{2}$ was employed in the previous example. This too produces an objective situation like the one investigated in example 4.3 of § 4, with $G=2S+1$, provided we let $\mathbb{Q}^{(\mathrm{app})}$ be some component of such a spin, say $\mathbb{S}^{z(\mathrm{app})}$, whose spectrum is equidistant $\{-S,\ldots,S\}$. Alternatively, one may apply an arrangement of G static external magnetic fields $\{\vec{H}_j\}$, all along the $\hat{k}$ direction, objectively producing the same transformation (by the Hamiltonians $\mathbb{H}_j=-\frac{1}{2}H_j\sigma^k$).

Think of the plane transverse to the $\hat{k}$ direction as the complex plane. As before, the component of $\vec{P}$ along $\hat{k}$ is unaffected. Starting at $t=0$ with a transverse component $P^{\perp}(0)$, one has, at time t, performed a concert of rotations giving

$$P^{\perp}(t)=\sum_{j=0}^{G-1}\xi_j\exp(-\mathrm{i}\omega_j t)P^{\perp}(0)$$

where $P^{\perp}$ is the transverse part of $\vec{P}$ represented as a complex number. A 'harmonious' concert of rotations is achieved if the rotation angles $\{\omega_j t\}$ are obtained from frequencies of the form $\omega_j=\omega j+\omega_0$. The criterion for having a measurement of the $\hat{k}$ aspect is therefore

$$\sum_{j=0}^{G-1}\xi_j\exp(-\mathrm{i}\omega_j t)=0$$

whereby the transverse component of $\vec{P}$ is brought to vanish and the resulting polarisation is the geometric projection $\vec{P}=\hat{k}(\hat{k}\cdot\vec{P})$.

First, let $\xi_j=1/G$ and $\omega_0=0$. Then

$$P^{\perp}(t)=\frac{1}{G}\sum_{j=0}^{G-1}[\exp(-\mathrm{i}\omega t)]^j P^{\perp}(0)=\frac{1}{G}\frac{1-\exp(-\mathrm{i}G\omega t)}{1-\exp(-\mathrm{i}\omega t)}P^{\perp}(0)$$

and the length is

$$|P^{\perp}(t)|=\frac{1}{G}\left|\frac{\sin(G\omega t/2)}{\sin(\omega t/2)}\right||P^{\perp}(0)|.$$

It is zero and the measurement condition is fulfilled at times $t_\nu=\nu 2\pi/G\omega$; ν are integers not divisible by G. At time $t_G=2\pi/\omega$ there is a Poincaré recurrence: $P^{\perp}(t_G)=P^{\perp}(0)$. This evolution in time is illustrated in figure 1(a).

Not only is the measurement condition satisfied repeatedly at instants that are closer in time the larger G is, but for a period of time, in between the start of the measurement and the recurrence, the object polarisation stays fairly close to the origin: $|P^{\perp}(t)|\simeq 0$. There are two foci: the point of measurement at the origin and $P^{\perp}(t_\mu)=(1/G)P^{\perp}(0)$, attained at time $t_\mu=\mu 2\pi/(G-1)\omega$; μ are integers not divisible by $(G-1)$. Throughout most of this intermediate period $|P^{\perp}(t)|\simeq(1/G)|P^{\perp}(0)|$, where it can thus be said that the measurement is approximately achieved, with a relative error of the order of $1/G$ which can be made small by design. In return for accepting such inaccuracy, the measurement becomes approximately permanent after a time of relaxation of the order of $2\pi/G\omega$.

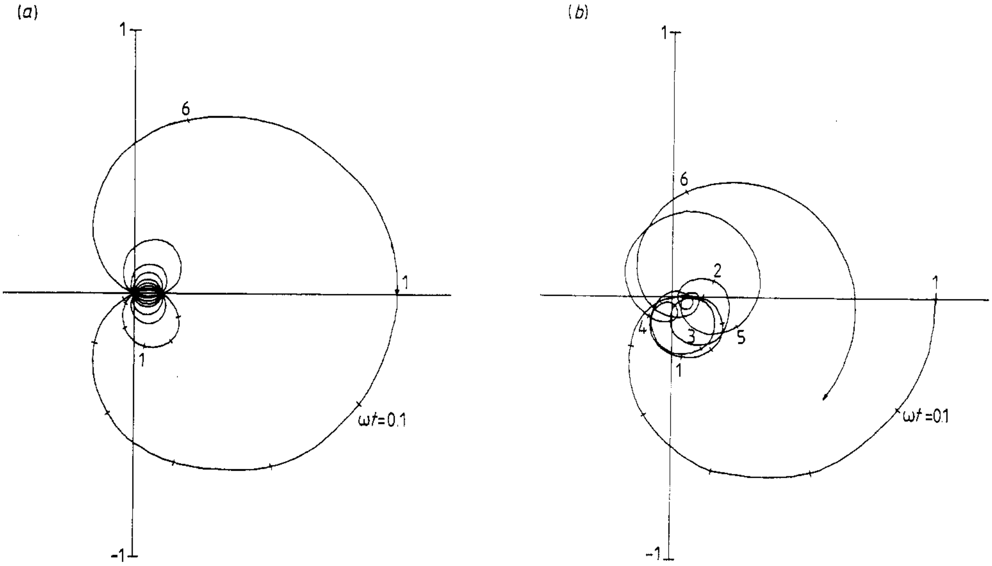

Figure 1. Motion, in the course of time, of the polarisation during a measurement on a spin-$\frac{1}{2}$ binary system. Shown is the track of the transverse polarisation vector in the plane perpendicular to the space $\hat{k}$ direction defining the aspect that is being measured, i.e. $P_\perp(t)$ as defined in § 5 when σ^k is measured using a cause with $G=10$ components. The measurement is complete at instances when the track crosses the origin $(0,0)$ and the polarisation has thereby been made to coincide with the $\hat{k}$ direction, its length being the geometric projection $\hat{k}\cdot\vec{P}(0)$. Scales are in units of initial transverse polarisation $P_\perp(0)$, starting at $(1,0)$. (a) Harmonious design ($\omega=1$), giving a periodic track with repeated crossings of $(0,0)$ and recurrence to the initial state $P^\perp=(1,0)$. (b) A design with a slight disharmony ($\delta\omega_j=0.12$, -0.07, 0.21, -0.15, -0.16, 0.10, 0.15, -0.13, 0.02, -0.09) causes the motion to become almost-periodic. The track cannot be counted on crossing the origin in finite time, but nevertheless stays essentially as close during the intermediate stage as it does in the perfectly harmonious design. With large G this would suffice for an approximately complete measurement, whose outcome remains essentially fixed at $(0,0)$ for a length of time.

For instance, in the Stern-Gerlach method one lets $G\to\infty$ and $\omega\to 0$, keeping $1/\tau=\omega G$ finite. Thereby

$$|P^\perp(t)|=\left|\frac{\sin(t/2\tau)}{t/2\tau}\right|\,|P^\perp(0)|\leqslant\frac{2\tau}{t}|P^\perp(0)|$$

and the Poincaré recurrence is put off into the infinite future.

The periodic recurrence ($G<\infty$) is notable. Shortly before it occurs, the purity of the state of the object starts to increase significantly and the state becomes as pure as it was to begin with (the object entropy decreases to its original value—but not below it, of course). So such a measurement transformation eventually unwinds itself.

Second, a slight 'disharmony', in the form of non-equidistant $\omega_j=\omega j+\delta\omega_j$, does not cause a drastic deterioration in the accuracy of the approximate measurement. As seen in figure 1(b), a set $\{\delta\omega_j\}$ of random disharmonies (here $|\delta\omega_j|\simeq 10\%$ of ω) causes the evolution to become almost-periodic. An almost-recurrence still occurs at $t=2\pi/\omega$, but the closer a recurrence one attends, the longer it takes before it happens. Apart from this, $P^\perp(t)$ still remains reasonably close to the origin at intermediate times, as it did in the harmonious design. However, the object polarisation 'never' actually crosses $P^{\perp}=0$, so, strictly speaking, the measurement criterion is not fulfilled in this design†. Nevertheless, it may come closer to a practical and feasible situation for a macroscopic apparatus; in which case one could imagine that $\mathbb{Q}^{(\mathrm{app})}$ was a 'random property' of the apparatus (i.e. an operator whose matrix is a member of a random-matrix ensemble for which the level spacing in the spectrum of eigenvalues has a most probable value (Brody *et al* 1981)). On the other hand, one cannot tolerate much laxity in the $\hat{k}$ directionality if the $\hat{k}$ aspect is to be measured sucessfully.

6. The probabilistic/statistical interpretation

The extension of the scope of quantum theory provided by objective causality helps to clarify a number of fundamental issues. We have already remarked upon the operational nature of properties, in connection with theorem 4.1(a). Let us not forego the opportunity to provide some further comments on the implications of (c) of that theorem.

Suppose that the initial object state is pure: $\mathbb{W}=\mathbb{P}_{\psi}=|\psi\rangle\langle\psi|$. It therefore corresponds to a state in which the ψth item of the 'greek' aspect $\mathscr{A}_{\mathrm{g}}$ is certain. The properties in $\mathscr{A}_{\mathrm{g}}$ have definite quantitative values and these values can be confirmed by means of a reproducing measurement $T\in\mathcal{M}(\mathscr{A}_{\mathrm{g}})$.

Despite this perfect certainty, which extends to every individual specimen (because the state is pure), the predictions concerning *different* aspects, say the 'latin' aspect $\mathscr{A}_{\mathrm{l}}$, are probabilistic. We predict $P(n)=\langle n|\mathbb{W}|n\rangle=|\langle n|\psi\rangle|^2$ for the nth item of $\mathscr{A}_{\mathrm{l}}$. If $P(n)<1$, which is generally the case, there is uncertainty (at least on the part of the theorist who makes the prediction) about the assignment of quantitative values to the properties in $\mathscr{A}_{\mathrm{l}}$. What is the physical significance of this uncertainty?

Perhaps the easiest response would be to say that the uncertainty represents an irreducible element of physical Nature. Then it needs no explanation. The uncertainty would be an intrinsic feature of every individual specimen.

Another response is suggested (but *not* implied) by the projection postulate. When $\mathscr{A}_{\mathrm{l}}$ is measured, in an ensemble of specimens forming a system, the nth item in $\mathscr{A}_{\mathrm{l}}$ occurs with frequency w_n and it is postulated that $w_n=P(n)$, the predicted probability. It might then be imagined that a definite value, A_n, of the property $\mathbb{A}\in\mathscr{A}_{\mathrm{l}}$ was attributed to each specimen, and was already present when the object is in the state $\mathbb{P}_{\psi}$. Then $P(n)$ would represent the distribution of values $\{A_n\}$ between the specimens of the system and the measurement by projection would detect it by some sort of filtering attuned to the aspect $\mathscr{A}_{\mathrm{l}}$. This viewpoint attempts to maintain classical realism and has been shown to lead to predictions (Bell inequalities) which both contradict standard quantum theory, and which do not agree with experiments designed to test them.

It thus appears that, while the probabilities $\{P(n)\}$ surely represent an objective feature associated with the object system (because they are defined exclusively in terms pertaining to the object), if they belong to individual specimens they cannot easily be given a statistical interpretation. This might lead one to believe that we need two kinds of probabilistic/statistical concepts in physics: the intrinsic uncertainty associated with the probabilistic interpretation of the quantities $\{P(n)\}$ in quantal systems and the

† An almost-periodic function, such as the exponential polynomial we are contemplating here, may—or may not—have zeros. To be more rigorous about this is beyond the scope of the present paper. Let it just be said that, due to the truncation in the decimal representation of the $\delta\omega_j$, there is a sense in which one can assert that it is most likely that this polynomial does not have zeros.

conventional statistical dispersion associated with classical random variables. If this is the case, then it requires a postulate, the projection postulate, to state that there is a process called 'measurement' which renders $\{P(n)\}$ observable as statistical frequencies $\{w_n\}$.

However, while the second of the responses above is too extremely statistical, one could say that the first response is too extremely probabilistic. We shall show that objective causality supplies a middle ground in which the two viewpoints can be reconciled, and even unified.

Historically the probability/statistics dichotomy was compounded in the way mixed states were originally defined. Namely, let $\{|\psi_j\rangle\}$ denote a set of pure quantum states pertaining to an ensemble of subsystems, wherein each subsystem state $\mathbb{P}_j = |\psi_j\rangle\langle\psi_j|$ occurs with weight ξ_j ($\Sigma_j\, \xi_j = 1$). Then the mixed state of the system is

$$\mathbb{W} = \sum_j \xi_j \mathbb{P}_j$$

and the probabilities predicted for the aspect $\mathscr{A}_1$ are

$$P(n) = \sum_j \xi_j P_j(n) \qquad P_j(n) = \langle n|\mathbb{P}_j|n\rangle = |\langle n|\psi_j\rangle|^2.$$

This expression would not contradict a different status to the quantal (conditional) probabilities $\{P_j(n)\}$ and the classical frequencies $\{\xi_j\}$.

But neither does it make the distinction compelling. Every operator $\mathbb{W} \in \mathscr{S}$ belongs to an aspect, say $\mathscr{A}_g$, so $\mathbb{W}$ can equally well be written in the form†

$$\mathbb{W} = \sum_\alpha W_\alpha \mathbb{P}_\alpha$$

where $\{W_\alpha\}$ are at the same time quantal probabilities $(P(\alpha) = \langle\alpha|\mathbb{W}|\alpha\rangle = W_\alpha)$ and statistical frequencies for the outcome of a reproductive measurement of $\mathscr{A}_g$ itself: $\mathbb{W} = \Sigma_\alpha\, \mathbb{P}_\alpha \mathbb{W} \mathbb{P}_\alpha$. On the other hand, there is no comparable way to extract the weights $\{\xi_j\}$ from a given $\mathbb{W}$, since the decomposition is not unique unless it is into orthonormal projectors, which $\{\mathbb{P}_\alpha\}$ are, but $\{\mathbb{P}_j\}$ are not.

For this reason the most prudent attitude is to maintain that the physical reality to be associated with the object in a state $\mathbb{W}$ resides *exclusively* in the aspect $\mathscr{A}_g$ defined by $\mathscr{L}_g = \{\mathbb{P}_\alpha\}$ and the statistical weights $\{W_\alpha\}$. This is the viewpoint adopted in the present work. It requires nothing but the classical logic of $\mathscr{A}_g$ and the classical statistics of $\{W_\alpha\}$, whether the state is pure ($W_\alpha = 1$ for $\alpha = \psi$, $W_\alpha = 0$ for $\alpha \neq \psi$) or not. So far there is no conceptual discrepancy between the quantal probabilities $\{P(\alpha)\}$ and the statistical weights $\{W_\alpha\}$. Neither is there any discrepancy *after* the measurement, since the new state $\mathbb{w} = \Sigma_n\, w_n \mathbb{P}_n$ can be interpreted along the same lines.

It remains to explain the physical meaning to be associated with the probabilities $\{P(n)\}$ which are predicted *before* the measurement, in particular the question of the uncertainty which they entail in a pre-measurement state in which, it is now claimed, the objective reality of the state $\mathbb{P}_\psi$ involves only classically certain and verifiable propositions.

Objective causality offers the following explanation. In order to actualise the aspect $\mathscr{A}_1$ to be measured, the object system has to be transformed physically. How this is to be done cannot be seen in the projection postulate. But with the scope extended

† See first footnote on page 4518 of this paper.

to the entire equivalence class of (physical) transformations which accomplish the measurement it is evident that the physical reality associated with $\{P(n)\}$ involves not only the object in its initial state $\mathbb{P}_\psi$, but the surroundings that eventually cause the transformation as well.

Thus, to interpret the predicted probabilities $\{P(n)\}$ one needs to take into account the whole situation in which the system *may* eventually find itself during the measurement of the aspect $\mathscr{A}_1$ for which the prediction is being made. Whereas the object *in itself* is in a state $\mathbb{P}_\psi$ whose reality is definite, those external influences introduce the element of uncertainty to be found in $\{P(n)\}$. Indeed, we have shown that every measurement with physical cause involves a range of different subsystem transformations (say $\mathbb{W} = \Sigma_j \xi_j \mathbb{W}_j$; $\mathbb{W}_j = \mathbb{U}_j \mathbb{W} \mathbb{U}_j^\dagger$). Thus it is necessary to treat different specimens differently, although they all start out in identical states $\mathbb{P}_\psi$ in order to achieve the actualisation of the new aspect $\mathscr{A}_1$ (assuming $\mathscr{A}_1 \neq \mathscr{A}_g$). We can understand the uncertainty of the predictions $\{P(n)\}$ as uncertainty about how each individual specimen is subsequently going to be influenced during the physical operations of the measurement process—and not as any intrinsic uncertainty in the reality of the initial state $\mathbb{P}_\psi$ of the object.

It is gratifying (indeed it is necessary) that every conceivable measurement transformation in $\mathscr{M}(\mathscr{A}_1)$ produces the same statistic: $w_n = P(n)$. This leaves no doubt as to how the uncertainty must be defined in terms of the initial state $\mathbb{P}_\psi$ *and* the aspect $\mathscr{A}_1$ which it is proposed to measure. It also shows that the predictions *have to* be probabilistic.

The uncertainty inherent in $\{P(n)\}$ pertains to a well defined hypothetical future situation in which the object possibly *may* find itself situated. Therefore the entire range of different (incompatible) aspects can be predicted. But these are experimental options: a choice has to be made by the investigator as to which aspect is actually going to be measured. Then, and only then, does the uncertainty enter into the realm of physical reality and become the statistical dispersion observed in the statistic $\{w_n\}$.

It is therefore quite appropriate, after all, to use the term 'uncertainty', since the predictor may be uncertain as to the course of hypothetical future events. But this need not imply any uncertainty as to the physical reality of the actual initial state $\mathbb{P}_\psi$†.

In fact, this uncertainty about the causes that influence the individual specimen seems necessary—it is a precondition for the measurement to work. Without getting in too deep at this point, let it be said by way of illustration that, in the measurement of spin by the Stern-Gerlach method (cf § 5 and the last paragraphs of the appendix) it is necessary, in order to achieve beam separation, that the incoming beam has a certain non-zero width, so that the individual specimens become situated in different magnetic fields (hence the inhomogeneity of the field in that method). This corresponds to the necessity of forming the convex combination $\Sigma_j \xi_j \mathbb{W}_j$ (traditionally 'an irreversible act of amplification') in order to obtain a state which is compatible with the measured aspect.

In this way it seems that the extended scope of objective causality allows us to unify the probabilistic and statistical viewpoints. The uncertainty in the predictions

† Of course, if the initial state $\mathbb{P}_\psi$ arises more or less spontaneously its aspect may be so intricate as to be beyond practical confirmation. The experimenter may lack the means to perform any of the measurement transformations in $\mathscr{M}(\mathscr{A}_g)$. This, however, is not a matter of principle. As we emphasised, it is *always* so that there is a physical reality associated with $\mathscr{A}_g$, whether or not it is practically accessible to measurement. The virtue of such aspects as $\mathscr{A}_1$ is that they can be chosen simple enough for the transformations in $\mathscr{M}(\mathscr{A}_1)$ to be within practical reach, by design, beforehand, and independently of $\mathbb{P}_\psi$.

$\{P(n)\}$ concerns not only the object system itself, but the entire situation in which it *may* become situated. The uncertainty need not be interpreted as inherent to each specimen, but rather provides a preview of events that may take place in the course of a future measurement. Every uncertain prediction $\{P(n)\}$ agrees with a possible future dispersed statistic $\{w_n\}$, $w_n = P(n)$, and this suffices for a satisfactory physical foundation of the probability calculus of quantum theory. We have shown that the agreement is a theorem, and need not be based on postulate.

Appendix. Some technical remarks

Let $\{|n\rangle | n \in \mathbb{I}\}$ be an orthonormal basis in $\mathbb{H}$. Then $\mathscr{L} = \{\mathbb{P}_n | n \in \mathbb{I}\}$ is a complete logical idea, where the projectors are orthogonal: $\mathbb{P}_n\mathbb{P}_{n'} = \delta_{nn'}\mathbb{P}_n$, with $\delta_{nn'} = 1$ if $n = n'$, and 0 if $n \neq n'$. It is easiest to assume that $\mathbb{H}$ is separable ($\mathbb{I}$ is countable) and, for simplicity, our notation is adapted to this case. However, this restriction is not necessary.

To generalise, one must understand by $\Sigma_n \mathbb{P}_n = \mathbb{1}$ a decomposition of the unity operator such that, for every $|\psi\rangle \in \mathbb{H}$, $\Sigma \{\mathbb{P}_n|\psi\rangle;\, n \in \mathbb{I}\} = |\psi\rangle$, where $\Sigma\{\,\cdot\,, n \in \mathbb{I}\}$ means (unconditional) convergence (in the norm $\|\cdot\|$ on $\mathbb{H}$) of the net defined by the directed set of finite subsets of $\mathbb{I}$ (ordered by inclusion) (Dunford and Schwartz 1963, Conway 1985). This decomposition is sometimes written as $\bigoplus_n \mathbb{P}_n = \mathbb{1}$. It can be generalised to projectors on any decomposition of $\mathbb{H}$ into orthogonal subspaces: $\mathbb{H} = \bigoplus_N \mathbb{H}_N$, where $\mathbb{H}_N = \mathbb{P}_N\mathbb{H}$ ($\bigoplus_N \mathbb{P}_N = \mathbb{1}$) are not necessarily one dimensional. We shall use this in the discussion of incomplete measurements in a subsequent paper.

Recall that, for a given $|\psi\rangle \in \mathbb{H}$, $\Sigma\{\mathbb{P}_n|\psi\rangle;\, n \in \mathbb{I}\} = \Sigma_{n=1}^{\infty} \langle n|\psi\rangle|n\rangle = |\psi\rangle$, where $\{|n\rangle$ $n \in \mathbb{N}\}$ is a countable orthonormal set (and a subset of a basis). But if $\mathbb{H}$ is not separable this orthonormal set need not be the same subset of a given basis for different $|\psi\rangle$.

Similarly, by $\mathbb{A} = \Sigma_n A_n\mathbb{P}_n$, if $\mathbb{H}$ is not separable, one must understand $\mathbb{A} = \Sigma\{A_n\mathbb{P}_n;\, n \in \mathbb{I}\} = \bigoplus_n A_n\mathbb{P}_n$, where the spectrum of $\mathbb{A}$ consists of eigenvalues: $\mathrm{sp}(\mathbb{A}) = \{A_n | n \in \mathbb{I}\}$. Thus, by definition, when $\mathbb{A}$ belongs to an aspect it is diagonalisable in terms of the corresponding complete logical idea $\mathscr{L}$.

We have chosen to define aspects (and ideas) in terms of simultaneous diagonalisation, rather than simultaneous spectral decomposition. Thus, for given $\mathscr{L}$:

$$\mathscr{A} = \{\mathbb{A} \,|\, \mathbb{A} = \textstyle\bigoplus_n A_n\mathbb{P}_n;\, \mathbb{P}_n \in \mathscr{L};\, A_n \in \mathbb{C};\, n \in \mathbb{I}\}$$

is the aspect based on $\mathscr{L}$, and every subset $\mathscr{I} \subseteqq \mathscr{A}$ is an idea. $\mathscr{L}$ is an idea, the complete logical idea of $\mathscr{A}$.

Every compact normal ($\mathbb{A}\mathbb{A}^\dagger = \mathbb{A}^\dagger\mathbb{A}$ for $\mathbb{A} \in \mathscr{B}(\mathbb{H})$) operator belongs to (at least) one aspect because it is diagonalisable (von Neumann 1955, Dunford and Schwartz 1963, Conway 1985). In particular, every member of $\mathscr{S}$ is compact and self-adjoint (hence normal). Therefore any state $\mathbb{W}$ belongs to an aspect.

Some normal operators may not belong to any aspect but may be compatible with one (or several). The reason for allowing this is that, since every state belongs to an aspect, any measurement result must have the form $\mathbb{w} = T(\mathbb{W}) = \Sigma_n w_n\mathbb{P}_n$. This means that the resulting statistic $\{w_n\}$ can only assign definite quantitative values to properties whose (eigen)values can be indicated unambiguously in terms of $n \in \mathbb{I}$, i.e. properties which are members of the same aspect as $\mathbb{w}$. It would be a consistent terminology to call such operators 'observables'.

Operators outside aspects would require 'coarse graining' in order to become observable. Since the discussion of this is the subject of a subsequent paper, we restrict the present remarks to an example which demonstrates the point.

Assume that $\mathbb{H}$ is separable. Let Ω be a finite interval of $\mathbb{R}$, the real numbers. Then there is a bounded normal operator ('position', say): $\mathbb{q} \equiv \int_\Omega q \, \mathrm{d}\mathbb{E}(q)$, where $\mathbb{E}$ is the spectral measure on Ω. In the usual way, if $\{|q)|q \in \Omega\}$ defines the position representation, then $\mathbb{q} = \int_\Omega \mathrm{d}q \, q|q)(q|$, with $(q|q') = \delta(q-q')$ and $\mathrm{Tr}(\mathbb{A}) = \int_\Omega \mathrm{d}q \, (q|\mathbb{A}|q)$, for $\mathbb{A} \in \mathscr{B}_1$ (the trace-class operators on $\mathbb{H}$). Let $\{\Omega_N | N \in \mathbb{N}\}$ be a partition of Ω. Then $\mathbb{E}(\Omega_N) = \mathbb{P}_N$ is the orthogonal projector on a subspace of $\mathbb{H}$: $\mathbb{P}_N \mathbb{H} = \mathbb{H}_N$, $\mathbb{H} = \bigoplus_N \mathbb{H}_N$, and $\Sigma_N \mathbb{P}_N = \mathbb{1}$.

Let $q_N \in \Omega_N$, define $\tilde{\mathbb{q}} \equiv \Sigma_N q_N \mathbb{P}_N$ and let $\mathrm{d}q_N$ be the extent of Ω_N (i.e. $\mathrm{d}q_N \equiv \sup_{q,q' \in \Omega_N}(q-q')$). Then $\tilde{\mathbb{q}}$ is a coarse graining of $\mathbb{q}$, with resolution $\{\mathrm{d}q_N\}$. $\tilde{\mathbb{q}}$ belongs to an aspect, namely any aspect $\mathscr{A}$ for which $\mathbb{P}_N = \Sigma_{n \in \mathscr{P}_N} \mathbb{P}_n$ for all N, where $\mathscr{P} = \{\not{n}_N | \not{n}_N \subseteqq \mathbb{I}, N \in \mathbb{N}\}$ is a partition of $\mathbb{I}$. Therefore $\tilde{\mathbb{q}}$ is observable in a (complete) measurement of $\mathscr{A}$. Furthermore, $\|\tilde{\mathbb{q}} - \mathbb{q}\| < \varepsilon$, where $\varepsilon \equiv \sup_N(\mathrm{d}q_N)$ (Dunford and Schwartz 1963). In this sense we can find an observable $\tilde{\mathbb{q}}$ which can be made arbitrarily close to $\mathbb{q}$ by fine graining ($\varepsilon \to 0$). If $P(q) \equiv (q|\mathbb{W}|q)$ is the probability density over Ω for $\mathbb{q}$, then the result of the measurement, the statistic $\{w_n\}$, gives

$$\tilde{w}_N = \sum_{n \in \not{n}_N} w_n = \mathrm{Tr}(\mathbb{W}\mathbb{P}_N) = \int_\Omega \mathrm{d}q \, (q|\mathbb{W}\mathbb{E}(\Omega_N)|q) = \int_{\Omega_N} \mathrm{d}q \, (q|\mathbb{W}|q)$$
$$= \int_{\Omega_N} \mathrm{d}q \, P(q).$$

Thus one can say that a measurement of $\mathscr{A}$ produces a coarse-grained statistic $\{\tilde{w}_N\}$, according to which $\tilde{\mathbb{q}}$ can be assigned values $\{q_N\}$ with resolution $\{\mathrm{d}q_N\}$. However, inside each interval Ω_N the choice of q_N is arbitrary and so one cannot claim to have assigned to $\mathbb{q}$ a *definite* value from its spectrum. But the resolution of $\mathbb{q}$ provided by the observable $\tilde{\mathbb{q}}$ can be made arbitrarily sharp.

Operators which do not belong to any aspect may still serve as causes of transformations. For instance, if $\mathbb{q}$ is position it may generate momentum 'boosts': $\mathbb{U} = \exp(\mathrm{i}P\mathbb{q}/\hbar)$ is the cause of a unitary transformation which translates the object through P in momentum.

In a wider context, the operator $\mathbb{Q}^{(\mathrm{app})}$ in example 2.4 may have a continuous spectrum, so that the operation $T(\mathbb{W}) = \Sigma_j \xi_j \mathbb{U}_j \mathbb{W} \mathbb{U}_j^\dagger$ means

$$T(\mathbb{W}) = \int \mathrm{d}Q \, \xi(Q) \mathbb{U}(Q) \mathbb{W} \mathbb{U}(Q)^\dagger$$

where

$$\mathbb{U}(Q) = \exp(-\mathrm{i}Q\mathbb{p}/\hbar) = {}^{(\mathrm{app})}(Q|\exp(-\mathrm{i}\mathbb{Q}^{(app)} \otimes \mathbb{p}/\hbar)|Q)^{(\mathrm{app})}$$

and $\xi(Q) = {}^{(\mathrm{app})}(Q|\mathbb{W}_{\mathrm{old}}^{(\mathrm{app})}|Q)^{(\mathrm{app})}$ is the initial probability density for the apparatus to supply the parameter Q. Since $\mathbb{Q}^{(\mathrm{app})}$ is not to be measured there is no reason to restrict the attention to an observable version. For this reason we do not restrict definition 2.2(b) to countable causes.

For instance, in the Stern–Gerlach method (cf § 5(c)) one can use the position of the particles transverse to the beam direction to record the spin-measurement result. Then, if $(Q|\psi\rangle$ is the wavefunction in the initial state $\mathbb{W}_{\mathrm{old}}^{(\mathrm{app})} = |\psi\rangle\langle\psi|$, the initial probability density across the aperture (the beam profile) is

$$\xi(Q) = {}^{(\mathrm{app})}(Q|\mathbb{W}_{\mathrm{old}}^{(\mathrm{app})}|Q)^{(\mathrm{app})} = |(Q|\psi\rangle|^2$$

and by field inhomogeneity Q determines the field the particle experiences. We thus have the $G \to \infty$ case due to the continuous position spectrum. The initial (transverse) position is not measured to better resolution than to establish the beam aperture. In fact, to measure the transverse position $Q^{(\mathrm{app})}$ at the entrance would prevent its use as a recording device. It would prevent beam separation by the indeterminacy in the transverse momentum which is the consequence of such a measurement.

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